Sheaves of modules

Monday, November 4, 2024 5:10 PM

start of 06.11.24 lec

$(X, \mathcal{O}_X)$ be a ringed space.

Notation. $\mathrm{Mod}\,\mathcal{O}_X$ = category of sheaves of $\mathcal{O}_X$-mods on X. =: category of $\mathcal{O}_X$-mods.

objects of $\mathrm{Mod}\,\mathcal{O}_X$ = sheaves of $\mathcal{O}_X$-mods on X

A morphism $f: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-mods is a map of sheaves s.t $f_{|U}: \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ lin for $U \underset{\text{open}}{\subseteq} X$. Such a morphism is called $\mathcal{O}_X$-linear.

e.g: $g: Y \to X$ morphism of schemes

$g_* \mathcal{O}_Y \in \mathrm{Mod}\,\mathcal{O}_X$, $g^\#: \mathcal{O}_X \to g_* \mathcal{O}_Y$ is $\mathcal{O}_X$-lin by definition.

§ Operation on $\mathrm{Mod}\,\mathcal{O}_X$.

- Given $\mathcal{O}_X$-linear $f: \mathcal{F} \to \mathcal{G}$, $\ker f$, $\operatorname{coker} f$, $\operatorname{im}(f)$ are in $\mathrm{Mod}\,\mathcal{O}_X$.

- $\mathrm{Mod}\,\mathcal{O}_X$ admits $\otimes_{\mathcal{O}_X}$ pdt.

$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) :=$ sheafification of the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

- Admits an internal hom sheaf

$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$

- $\mathrm{Mod}\,\mathcal{O}_X$ admits arbitrary direct sum and product

Ex: Check $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$ in $\mathrm{Mod}\,\mathcal{O}_X$.

- i) $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map of ringed spaces. For $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}_Y$, $f_* \mathcal{F}$ is naturally an $\mathcal{O}_X$-mod: $f_* \mathcal{F}$ is a $f_* \mathcal{O}_Y$ module; Then $f_* \mathcal{O}_Y$ is an $\mathcal{O}_X$-mod via the map $f^\#: \mathcal{O}_X \to f_* \mathcal{O}_Y$.

- ii) $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ map of ringed spaces.

$f^\#: \mathcal{O}_X \to f_* \mathcal{O}_Y$.

Def. $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ map of ringed spaces, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. The $f^*\mathcal{G}$ = pull back of $\mathcal{G}$ via f is $f^{-1}\mathcal{G} \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$

[$\mathcal{O}_X \to f_*\mathcal{O}_Y$, gives $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$, The $\otimes$ is taken on the ringed space $(Y, f^{-1}\mathcal{O}_X)$

Rmk. 1) $f^*\mathcal{G}$ ~~is~~ $\in \mathrm{Mod}_{\mathcal{O}_Y}$ (Ex).

2) $(Y, \mathcal{O}_Y)$, $(X, \mathcal{O}_X)$ locally ringed spaces. (eg schemes).

$$(f^*\mathcal{G})_y = \mathcal{G}_{f(y)} \otimes_{\mathcal{O}_{X, f(y)}} \mathcal{O}_{Y,y}$$

Recall. $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$ ~~gi~~ induces $\mathcal{O}_{X, f(y)} \to \mathcal{O}_{Y,y}$.

§ $\tilde{\ }$ construction.

Let A be a ring, $M \in \mathrm{Mod}_A$. Consider the presheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

$$\tilde{M}(U) = \left\{ \text{set maps } s: U \to \coprod_{p \in U} M_p \;\middle|\; \begin{array}{l} \text{(i) } s(p) \in M_p \\ \text{(ii) for } p \in U, \exists\, p \in V \subseteq_{\text{open}} U \\ \text{and } m \in M \text{ s.t.} \\ s(q) = m/f \in M_q \\ \text{for some } f \notin \bigcup_{q \in V} q \end{array} \right\}$$

Thm (Prop 5.1, Hart)

a) $\tilde{M}$ is an $\mathcal{O}_X$-mod.

b) For $p \in \mathrm{Spec}(A)$, $(\tilde{M})_p \xrightarrow{\sim} M_p$ as A_p modules

c) For $f \in A$, The natural map $M_f \xrightarrow{\sim} \tilde{M}(D(f))$ is an isom.

d) In particular $\Gamma(\mathrm{Spec}(A), \tilde{M}) \leftarrow M$ is an isom.

Pf. (a), (b) clear, note (c) $\Rightarrow$ (d)

(c) Lemma: Let $s \in \Gamma(\tilde{M}, \mathrm{Spec}(A))$ such that $s|_{D(g)} = 0$ for some $g \in A$. Then $g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

[illegible] g_i such that $s = m_i/g_i$ on $D(g_i)$

$g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

Pf. Choose $g_1, g_2, \ldots g_r$ such that $s = m_i/g_i$ on $D(g_i)$ and $\bigcup_{i=1}^{r} D(g_i) = \operatorname{Spec}(A)$.

Since $s|_{D(g_i) \cap D(g)} = s|_{D(g_i g)} = 0$

$m_i/g_i = 0 \in M_{\mathfrak{q}} \quad \mathfrak{q} \in D(g_i g)$

$\Rightarrow \quad m_i/g_i = 0 \in M_{g_i g} = M_{g_i}[1/g]$

$\Rightarrow \quad g^{t_i} \cdot \frac{m_i}{g_i} = 0$ in A_{g_i} for some t_i

$\Rightarrow \quad g^{t_i} \cdot s|_{D(g_i)} = 0$

Take $m = \max\{t_1, t_2 \ldots t_r\}$.

Then $g^m \cdot s|_{D(g_i)} = 0 \ \forall i \Rightarrow g^m \cdot s = 0$ ∎

Thm, c Given $s \in \tilde{M}(D(f))$, choose $g_1, g_2, \ldots, g_r$ such that
- $D(f) = \bigcup_{i=1}^{r} D(g_i)$ and
- On $D(g_i)$, $s = m_i/g_i$ for some $m_i \in M$.

i.e. $g_i s = \frac{m_i}{1}$ on $D(g_i)$

By Lemma $\quad g_i^{n_i+1} s. = g_i^{n_i} \cdot m_i \in \Gamma(\operatorname{Spec} A[1/f], \tilde{M})$

for some n_i $\quad (*)$

$$\bigcup_{i=1}^{r} D(g_i) = \bigcup_{i=1}^{r} D(g_i^{n_i+1}) = D(f)$$

$\Rightarrow \quad V(f) = V(g_1^{n_1+1}, g_2^{n_2+1} \ldots, g_r^{n_r+1})$

$\Rightarrow \quad f \in \sqrt{(g_1^{n_1+1} \ldots g_r^{n_r+1})}$

$\Rightarrow \quad f^m = g_1^{n_1+1}\beta_1 + \cdots + g_r^{n_r+1}\beta_r$ for some m & $\beta_1 \ldots \beta_r \in A$.

in $\Gamma(D(f), \tilde{M})$

$$(*) \quad \Rightarrow f^m s = \left(\sum g_i^{n_i+1}\beta_i\right) s \overset{\downarrow}{=} \sum g_i^{n_i}\beta_i \cdot m_i \in M$$

$\Rightarrow \quad s \in \operatorname{Im}(M_f \longrightarrow \tilde{M}(D(f)))$.

Rmk. Given $M \in \operatorname{Mod}_A$, consider the presheaf $\mathcal{M}$ of $\mathcal{O}_X$ modules where $X = \operatorname{Spec} A$:

$\mathcal{M}(V) = \{ m/f \mid f \notin \cup \mathfrak{q}, \mathfrak{q} \in V \}$.

of $\mathcal{O}_X$ modules.

$\mathcal{M}(V) = \{ m/f \mid f \notin \mathfrak{q}, \mathfrak{q} \in V \}$.

Then $\tilde{M}$ is the sheafification of $\mathcal{M}$.

Thm. Given an A-mod map $f : M \to N$, one naturally gets an $\mathcal{O}_{\operatorname{Spec} A}$ linear map $\tilde{f} : \tilde{M} \to \tilde{N}$. Moreover $\widetilde{id_M} = id_{\tilde{M}}$ and $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

So $M \mapsto \tilde{M}$ is a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules.

Thm. The functor $M \mapsto \tilde{M}$ above has the following properties.

(i) The natural map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}(A)}}(\tilde{M}, \tilde{N})$ is an isom.

(ii) $X = \operatorname{Spec} A$, $\mathcal{F} \in \operatorname{Mod}_{\mathcal{O}_X}$, $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \to \operatorname{Hom}_A(M, \Gamma(X, \mathcal{F}))$ is an isom. Conversely for $\mathcal{G} \in \operatorname{Mod}_{\mathcal{O}_X}$, if the natural map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \xrightarrow{\psi^{\mathcal{G}}_{\mathcal{F}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), \Gamma(X, \mathcal{F}))$ is an isomorphism, then $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ for some $M \in \operatorname{Mod}_A$.

(iii) $\widetilde{\left(\bigoplus_{i \in I} M_i \right)} = \bigoplus_{i \in I} \tilde{M}_i$ in $\operatorname{Mod}_{\mathcal{O}_{\operatorname{Spec}(A)}}$.

(iv) $\widetilde{M \otimes_A N} \xrightarrow{\sim} \tilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec}(A)}} \tilde{N}$.

(v) $\varphi : A \to B$ ring homo, $M \in \operatorname{Mod}_A$, $(\varphi^{\#})^*(\tilde{M}) \xrightarrow{\sim} \widetilde{M \otimes_A B}$ in $\operatorname{Mod}_{\operatorname{Spec} B}$

(vi) φ as in (v), $N \in \operatorname{Mod}_B$, $(\varphi^{\#})_* \tilde{N} = \tilde{N}$ where on the right N is considered as an A-mod via restriction of scalars.

Pf (ii) sufficient cond^n ensuring $\mathcal{G} \xrightarrow{\sim} \tilde{M}$.

Take $M = \Gamma(X, \mathcal{G})$. Our assumption gives a map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \tilde{M}) \xrightarrow[\sim]{\psi^{\mathcal{G}}_{\tilde{M}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), M)$ that $\psi^{\mathcal{G}}_{\tilde{M}}(\varphi_1) = id_M$.

$\varphi_1 : \mathcal{G} \to \tilde{M}$ such that $\psi^{\mathcal{G}}_{\tilde{M}}(\varphi_1) = id_M$.

The isom $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\psi^{\tilde{M}}_{\mathcal{G}}} \operatorname{Hom}_A(M, M)$ gives

$\varphi_1 : \mathcal{G} \to M$ such ...

The isom $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\Psi^{\tilde{M}}_{\mathcal{G}}} \mathrm{Hom}_A(M, M)$ gives $\varphi_2 : \tilde{M} \to \mathcal{G}$ such that $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = id_M$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \arrow[r, "\Psi^{\tilde{M}}_{\mathcal{G}}"] \arrow[d, "\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \varphi_1)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_A(M, M) \arrow[d, "\mathrm{Hom}(M, id) = id"] \\
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{M}) \arrow[r, "\Psi^{\tilde{M}}_{\tilde{M}}"'] & \mathrm{Hom}(M, M)
\end{tikzcd}$$

So $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = \Psi^{\tilde{M}}_{\tilde{M}}(\varphi_1 \cdot \varphi_2) = id$

Since $\Psi^{\tilde{M}}_{\tilde{M}}$ is an isom $\varphi_1 \cdot \varphi_2 = id$.

Thm. $M, N \in \mathrm{Mod}_A$. $f : M \to N$ A-linear.

Then $\ker \tilde{f} \simeq \widetilde{\ker f}$, $\widetilde{\mathrm{coker}\, f} \simeq \mathrm{coker}\, \tilde{f}$

Pf. Have $\ker f \to M \to N$. This gives a complex $\widetilde{\ker f} \to \tilde{M} \to \tilde{N}$. This gives a map $\widetilde{\ker f} \to \ker \tilde{f}$, which is an isom at every stalk.

Thm. $0 \to M' \to M \to M'' \to 0$ is exact in Mod_A

$\iff$ $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact in $\mathrm{Mod}_{\mathcal{O}_X}$.

§ Quasi coherent (QCoh) sheaves (ref stacks project tag 01BD)

$(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$

Def. $\mathcal{F}$ is QCoh if for every $x \in X$, $\exists$ a nbhd U of x and an exact seq

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

Thm. $X = \mathrm{Spec}(A)$, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is QCoh iff $\mathcal{G} \simeq \tilde{M}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf. $\Leftarrow$ Choose a presentation $\bigoplus_{j \in J} A \to \bigoplus_{i \in I} A \to M \to 0$

This gives an exact seq $\bigoplus_{j \in J} \mathcal{O}_X \to \bigoplus_{i \in I} \mathcal{O}_X \to \tilde{M} \to 0$.

$\Rightarrow$ Choose $f_1, f_2, \ldots f_n \in A$ such that $X = \bigcup_{j=1}^{n} D(f_j)$ and for each j, have an exact seq

$\bigoplus \mathcal{O} \ldots \to \bigoplus \mathcal{O} \ldots \to \mathcal{G}|_{D(f_j)} \to 0$

exact seq

$$\bigoplus_{J} \mathcal{O}_{D(f_j)} \longrightarrow \bigoplus_{I} \mathcal{O}_{D(f_j)} \longrightarrow \mathcal{G}|_{D(f_j)} \longrightarrow 0$$

Since $\mathcal{O}_{D(f_j)} = \widetilde{A[1/f_j]}$, They are Qcoh.

So The coker $\mathcal{G}|_{D(f_j)} \xrightarrow{\sim} \widetilde{\Gamma(D(f_j), \mathcal{G})}$

Let $i_j : D(f_j) \hookrightarrow X$, $i_{j_1 j_2} : D(f_{j_1}) \cap D(f_{j_2}) \hookrightarrow X$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad || $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D(f_{j_1} f_{j_2})$

be The natural open immersions

We have an exact seq

$$0 \to \mathcal{G} \longrightarrow \bigoplus_{j} (i_j)_* (\mathcal{G}|_{D(f_j)}) \longrightarrow \bigoplus (i_{j_1 j_2})_* (\mathcal{G}|_{D(f_{j_1} f_{j_2})})$$

$$\begin{array}{ccc} \mathcal{G} \cong \widetilde{\ } & & \mathcal{G}\cong \widetilde{\ } \\ \Gamma(D(f_j), \mathcal{G}) \text{ on } X & & \Gamma(D(f_{j_1} f_{j_2}), \mathcal{G}) \text{ on } X \\ (s_j) & \longmapsto & ((s_{j_1} - s_{j_2})) \end{array}$$

$\mathcal{G}$ being The kernel, is also Qcoh.

• End of 6.11.24 lec

Thm: Let X be a scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$.

(i) $\mathcal{G}$ is Qcoh iff for every affine open $U \subseteq X$, $\mathcal{G}|_U \xrightarrow{\sim} \widetilde{M}_U$ for some $M_U \in \mathrm{Mod}_{\mathcal{O}_X(U)}$.

(ii) $\mathcal{G}$ is Qcoh iff $\exists$ an affine open covering $X = \cup U_\lambda$ s.t $\mathcal{G}|_{U_\lambda} \xrightarrow{\sim} \widetilde{M}_\lambda$ for some $M_\lambda \in \mathrm{Mod}_{\mathcal{O}_X(U_\lambda)}$

(iii) When X is affine checking $\mathcal{G} \cong \widetilde{M}$ iff $\mathcal{G}|_{U_\lambda} \cong \widetilde{M}_\lambda$ for some affine cover $X = \cup U_\lambda$ and $M_\lambda \in \mathrm{Mod}\, \mathcal{O}_X(U_\lambda)$.

Thm. X be a scheme, The category of Qcoh sheaves $\mathrm{Qcoh}(X)$ is abelian, admits arbitrary direct sum.

Pf. HW:

§ Coherent Sheaves: (abv. coh)

The notion of coh sheaves on a scheme X is defined under The additional assumption X is ... additionally

- The notion of coh sh... defined under the additional assumption X is locally noetherian (be careful, Hart assumes additionally X is noeth, which we do not).

<u>Def.</u> Let X be a locally noetherian scheme. $\mathcal{G} \in \mathrm{Mod}\,\mathcal{O}_X$. $\mathcal{G}$ is called <u>coherent</u> (ab coh) if one of the following equivalent cond^n are satisfied (eq. is HW)

(i) $\mathcal{G}$ is Qcoh and for every affine open $U \subseteq X$, $\Gamma(U, \mathcal{G})$ is a finitely generated $\mathcal{O}_X(U)$ mod.

(ii) There is an affine open covering $X = \bigcup_\lambda U_\lambda$ s.t $\mathcal{G}|_{U_\lambda}$ is Qcoh and $\Gamma(U_\lambda, \mathcal{G})$ is a f.g $\mathcal{O}_X(U_\lambda)$ mod for $\forall \lambda$.

(iii) For every affine open $U \subseteq X$ or on each element U_λ of an affine open covering $X = \cup U_\lambda$, $\mathcal{G}|_U$ or $\mathcal{G}|_{U_\lambda}$ is isom to $\tilde{M}$ for some f.g $\mathcal{O}_X(U)$ or $\mathcal{O}_X(U_\lambda)$-mod M.

<u>Thm.</u> $\mathrm{QCoh}(X)$ is closed under taking ker, coker, $\oplus$, $\otimes$, $\mathrm{QCoh}(X)$ is an abelian category.

<u>Notation</u> $\mathrm{Coh}(X)$ = set of all coh $\mathcal{O}_X$-mods on X.

<u>Thm.</u> X, Y be noeth schemes.

(i) $\varphi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin map between coh sheaves $\ker\varphi$, $\mathrm{coker}\,\varphi$ coh.

(ii) $f : Y \to X$, $\mathcal{F} \in \mathrm{Coh}(X)$, $f^*\mathcal{F} \in \mathrm{Coh}(Y)$.

(iii) If f is finite $\mathcal{G} \in \mathrm{Coh}(Y)$, $f_*\mathcal{G} \in \mathrm{Coh}(X)$.

(iv) $\psi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin $\mathcal{G} \in \mathrm{QCoh}(X)$, $\mathcal{F} \in \mathrm{Coh}(X)$. $\mathrm{Im}\,\psi \in \mathrm{Coh}(X)$.

(iv) ...
   $\mathrm{Im}\,\varphi \in \mathrm{Coh}(X)$.
(v) $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, so is $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$.

<u>Thm.</u> $f : X \to Y$ map of schemes, f quasi-compact and separated

&

(i.e inverse image of any quasi-compact subset is quasi compact or for some affine open cover $Y = \cup U_\lambda$, $f^{-1}(U_\lambda)$ is quasi-compact}; when X is noetherian, any f is automatically q. compact).

$\mathcal{F} \in \mathrm{QCoh}(X)$, $f_*\mathcal{F} \in \mathrm{QCoh}(Y)$.

<u>Pf.</u> W.L.O.G Y is affine. Choose a finite affine covering $f^{-1}(Y) = \bigcup_{j=1}^{r} U_j$

As X is separated, $U_{j_1} \cap U_{j_2}$ is affine

We have an exact seq

$$0 \to \mathcal{F} \to \oplus (i_j)_*(\mathcal{F}|_{U_j}) \longrightarrow \bigoplus_{j_1, j_2} (i_{j_1,j_2})_*(\mathcal{F}|_{U_{j_1} \cap U_{j_2}})$$
$$(s_j) \longmapsto (s_{j_1} - s_{j_2})$$

Applying f_* <u>get an exact seq</u>

$$0 \to f_*\mathcal{F} \to \oplus f_*(i_j)_*(\mathcal{F}|_{U_j}) \longrightarrow \oplus f_*(i_{j_1,j_2})_*\mathcal{F}|_{U_{j_1} \cap U_{j_2}}$$

$f_*(i_j)_* = (f \circ i_j)_*$ $f \circ i_j : U_j \longrightarrow Y$, both U_j, Y are affine, so the 2nd and 3rd terms are QCoh. So the kernel is QCoh.

Sheaves on Proj

Thursday, November 7, 2024 7:49 PM

Let $R = \bigoplus_{j\in\mathbb{N}} R_j$ be an $\mathbb{N}$-graded ring. We constructed a scheme $(X, \mathcal{O}_X) = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$; Set $R_+ = \bigoplus_{j>0} R_j$

Recall $\mathrm{Proj}(R) = \{ p \mid p$ homogeneous prime, $p \not\supseteq R_+ \}$

The top on $\mathrm{Proj}(R)$ has open basis $\{D_+(f)\}$ f homo of +ve deg

$D_+(f) = \{ p \in \mathrm{Proj}(R) \mid f \notin p \}$

$\mathcal{O}_X(D_+(f)) \cong (R[1/f])_0$.

§ Recall, a $\mathbb{Z}$-graded R mod is $M \in \mathrm{Mod}_R$ such that $M = \bigoplus_{\lambda\in\mathbb{Z}} M_\lambda$ as an ab group

and $R_{\lambda_1} \cdot M_{\lambda_2} \subseteq M_{\lambda_1+\lambda_2}$, $\forall \lambda_1 \in \mathbb{N}, \lambda_2 \in \mathbb{Z}$.

- A graded R-lin map (or simply a graded map) between two graded mods is an R lin map $\varphi: M \to N$ s.t $\varphi(M_\lambda) \subseteq N_\lambda$ $\forall \lambda\in\mathbb{Z}$.
- The cat of $\mathbb{Z}$-graded R-mods with graded R-lin maps forms a cat denoted Mod_R^{gr}

Given $M \in \mathrm{Mod}_R^{gr}$, we construct a sheaf of $\mathcal{O}_X$-mods, denoted $\tilde{M}$ on $X = \mathrm{Proj}\ R$

Caution: This $\tilde{M}$ is not the q coh sheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

- Note given a mult closed set S of homogeneous elts of R, $S^{-1}R$ is a $\mathbb{Z}$-graded ring. $S^{-1}M$ is a $\mathbb{Z}$-graded mod / $S^{-1}R$.

$$(S^{-1}R)_\lambda = \{ r/s \in S^{-1}R \mid r, s \text{ homo},\ \deg r - \deg s = \lambda \}$$

$$(S^{-1}M)_\lambda = \{ m/s \in S^{-1}M \mid m, s \text{ homo},\ \deg m - \deg s = \lambda \}$$

- For $p \in \mathrm{Proj}(R)$, set $S_p =$ homo elts of $R \setminus p$

$$M_{(p)} = (S_p^{-1}M)_0$$

Def of $\tilde{M}$: Consider the presheaf of $\mathcal{O}_X$-mods on X;

$\tilde{M}(U) = \{$ set maps $s: U \to \bigsqcup_{p\in U} M_{(p)} \mid$
- $s(p) \in M_{(p)}$
- For every U, $\exists$ a open nbhd $V \subseteq U$ of p, a homo $f \notin \bigcup_{q\in V} q$ & $m \in M$ s.t $s(q) = m/f \in M_{(q)} \}$

It's implicit that $\deg m = \deg f$

Recall: $\tilde{R} = \mathcal{O}_X$, So for $s \in \mathcal{O}_X(U)$, $t \in \tilde{M}(U)$,
$(st)(q) = \underset{R_{(q)}}{s(q)} \cdot \underset{M_{(q)}}{t(q)} \in M_{(q)}$ $\forall q \in U$

Thm: 1) $\tilde{M}$ is a sheaf for any $M \in \mathrm{Mod}_R^{gr}$.
For $p \in \mathrm{Proj}\,R$, the natural map $M_{(p)} \to (\tilde{M})_p$ is an isom with inverse given by $s \mapsto s(p)$ for a section $s \in \tilde{M}(U)$, $p \in U$.

2) Given a graded R-lin map $\varphi: M \to N$, naturally have $\tilde{\varphi}: \tilde{M} \to \tilde{N}$
for $t \in \tilde{M}(U)$, $p \in U$: $\tilde{\varphi}_U(t)(p) = \varphi_{(p)}(t(p))$; $\varphi_{(p)}: M_{(p)} \to N_{(p)}$

3) $\widetilde{\varphi_1 \cdot \varphi_2} = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$, $\widetilde{id_M} = id_{\tilde{M}}$

4) So $M \mapsto \tilde{M}$ is gives a functor $\mathrm{Mod}_R^{gr} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Thm: For $M \in \mathrm{Mod}_R^{gr}$; $\tilde{M} \in \mathrm{QCoh}(X)$.
Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for

Thm. For $M \in \mathrm{Mod}_R^{gr}$; $\tilde{M} \in \mathrm{Qcoh}(X)$.
Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for every homo elt f of +ve deg.

Pf. Since being qcoh is a local property, it's enough to check that for any homo elt f of of +ve deg $\tilde{M}_{|D_+(f)}$ is qcoh.

Set $\Gamma = M[1/f]_0$. The identity map $\Gamma \to \Gamma(D_+(f), \tilde{M})$ gives an $\mathcal{O}_{D_+(f)}$-line map
$\star$ $\tilde{\Gamma} \longrightarrow \tilde{M}_{|D_+(f)}$ as $\tilde{\Gamma}$ is qcoh on the affine $D_+(f)$.

Claim. $\star$ is an isom.

Pf. We check isom at the stalks at $p \in D_+(f)$.
Recall that $D_+(f) \xrightarrow{\sim} \mathrm{Spec}(R[1/f]_0)$
$q \longmapsto qR[1/f] \cap R[1/f]_0 = q_a$

The map induced by $\star : (M[1/f]_0)_{p_a} = \tilde{\Gamma}_{p_a} \longrightarrow \tilde{M}_p \xrightarrow{\sim} M_{(p)}$ $\star\star$

$$g \in S_p \quad , \quad \frac{m/f^a}{g/f^{e_2}} \longmapsto \frac{m f^{e_2}}{g f^{a}}$$

Note $(M[1/f]_0)_{p_a} = [(S_p)^{-1} M[1/f]]_0 = (S_p^{-1}M)_0$

So $\star\star$ is an isom.

Rmk. So $\tilde{M}$ can be thought to be obtained by gluing $\widetilde{M[1/f]_0}$ on $D_+(f)$ for different f's.

Thm. 1) $f: M \to N$ be a graded R-mod map.
$\ker \tilde{f} \xleftarrow{\sim} \widetilde{\ker f}$, $\mathrm{coker}\, \tilde{f} \xrightarrow{\sim} \widetilde{\mathrm{coker} f}$

2) If $0 \to M' \to M \to M'' \to 0$ is exact in Mod_R^{gr} [This includes the hypothesis that all the maps are graded]
Then $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is also exact in $\mathrm{Mod}_{\mathcal{O}_X}$

3) $(\bigoplus_{i \in I} M_i)^{\sim} \xleftarrow[\text{in } \mathrm{Mod}_{\mathcal{O}_X}]{\sim} \bigoplus_{i \in I} \tilde{M}_i$ for any set I.

Thm. 1) $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $M[1/f]_0 = 0$
$\iff$ for a collection of homo elt of +ve deg $f_1, f_2, \dots, f_r$ s.t $X = \bigcup_{i=1}^{r} D_+(f_i)$ and and any $m \in M_n$ where $\deg f_i \mid n \ \forall i$,
$\exists\, n_1, n_2, \dots n_r$ s.t $f_i^{n_i} \cdot m = 0$

[A special case is when one can choose f_i's of deg 1 s.t $X = \bigcup_{i=1}^{r} D(f_i)$]

2) For $M \in \mathrm{Mod}_R^{gr}$,
$i: \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow M$. $\tilde{i}$ is an isom.

Pf. $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $\tilde{M}_{|D_+(f)} = 0$
$\underset{\tilde{M}\text{ qcoh}}{\iff}$ ———————" ——— $M[1/f]_0 = 0$
$\iff$ for a collection $\{f_i\}_{i \in I}$, f_i homo of +ve deg s.t $X = D_+(f_i)$, $M[1/f_i]_0 = 0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

s.t $X = D_+(f_i)$, $M[1/f_i]_0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

$M[1/f_i]_0 = \{ m/f_i^n \mid \deg m = n \deg f_i \}$

$M[1/f_i]_0 = 0 \iff$ for $m \in M_\lambda$ s.t $\deg f_i \mid \lambda$
$m/f_i^{\lambda/\deg f_i} = 0 \in M[1/f_i]_0 \subseteq M[1/f_i]$

$\iff f_i^{n_i} m = 0$ for some n_i.

2) Consider the exact seq in Mod_R^{gr}

$$0 \to \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \to \overset{M}{\overset{\shortparallel}{\bigoplus_{\lambda \in \mathbb{Z}}}} M_\lambda \to Q \to 0$$

Every nonzero ~~el~~ homo elt in Q lifts to a non-zero homo elt, m, of $-$ve deg in M

Now for every homo elt, f, of R of $+$ve deg

$\deg f^n . m \geq 0$ for $n >> 0 \Rightarrow f^n . \underset{\text{im in } Q}{\bar{m}} = 0$

$\Rightarrow \tilde{Q} = 0$

$\Rightarrow \quad 0 \to \widetilde{\bigoplus_{\lambda \in \mathbb{N}} M_\lambda} \to \tilde{M} \to 0$ is exact.

End of 08.11.24 Lec

§. $\mathcal{O}_X(n)$, $\mathcal{F}(n)$, . . . $X = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$

Def. • For $M \in \mathrm{Mod}_R^{gr}$, $M(n) \in \mathrm{Mod}_R^{gr}$ is the object whose underlying R-mod is M, but

$$M(n)_m = M_{n+m}$$

• $\mathcal{O}_X(n) := \widetilde{R(n)}$ • For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

$\mathcal{F}(n)$ is called the n-th Serre twist of $\mathcal{F}$.

Note There is a map $M_0 \to \Gamma(X, \tilde{M})$ and so
$M_n = (M(n))_0 \longrightarrow \Gamma(X, \widetilde{M(n)})$

Def: Y scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_Y}$ is called loc free if $\exists$ an open covering $Y = \bigcup_{\alpha \in I} U_\alpha$ s.t $\mathcal{F}|_{U_\alpha} \simeq \bigoplus_{g_\alpha} \mathcal{O}_{U_\alpha}$ $g_\alpha \in \mathbb{N}$ in $\mathrm{Mod}_{\mathcal{O}_{U_\alpha}}$

Prop • There are natural maps $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$ and $\mathcal{F}(n) \otimes \mathcal{O}_X(m) \to \mathcal{F}(n+m)$.

• Suppose $\deg f = m > 0$, Then

(i) $\mathcal{O}_{D_+(f)} \longrightarrow \mathcal{O}_X(m)|_{D_+(f)}$ is an isom in $\mathrm{Mod}_{\mathcal{O}_{D_+(f)}}$
$1 \longmapsto f/1$

(ii) $\mathcal{F}(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(n+m)|_{D_+(f)}$

Pf. (i) Using qcoh, enough to prove

$$\begin{array}{ccc} \Gamma(D_+(f), \mathcal{O}_X) & \longrightarrow & \Gamma(D_+(f), \mathcal{O}_X(m)) \\ \shortparallel & & \uparrow S \\ R[1/f]_0 & \longrightarrow & [R(m)[1/f]]_0 = (R[1/f])_m \\ 1 & \longmapsto & f/1 \end{array}$$

is an isom

$R[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{O})$
$1 \longmapsto f/1$
is an isom, inverse $s/f \longleftarrow s$

(ii) Since $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$, enough to prove $\mathcal{O}_X(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(n+m)|_{D_+(f)}$ is an isom

q-coh $\Rightarrow$ $\mathcal{A}$ enough to check the isom $\widetilde{x} \otimes \widetilde{y} \longmapsto xy$

$$\begin{array}{ccc} R[1/f]_n \otimes_{R[1/f]_0} R[1/f]_m & \longrightarrow & R[1/f]_{n+m} \\ \downarrow\wr & & \| \\ R[1/f]_n \otimes_{R[1/f]_0} f \cdot R[1/f]_0 & \xrightarrow{\sim} & f \cdot R[1/f]_n \end{array}$$

§ Given $\mathcal{F} \in QCoh(X)$, $X = Proj(R)$ we want to construct a candidate mod M s.t. $\widetilde{M} \xrightarrow{\sim} \mathcal{F}$.

Caution. Such an M need not exist.

Def. Given $\mathcal{F} \in Mod_{\mathcal{O}_X}$, set $\Gamma_* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$
$\Gamma_* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)$

Prop
- $\Gamma_* \mathcal{F}$ is naturally a $\Gamma_* \mathcal{O}_X$ mod
- There is a map of graded rings $R \to \Gamma_* \mathcal{O}_X$.
- Thus $\Gamma_* \mathcal{F}$ is naturally an R-mod.
- There is an $\mathcal{O}_X$-lin map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$

Pf
- $\Gamma_* \mathcal{O}_X = \bigoplus_{n \in \mathbb{N}} \Gamma(X, \mathcal{O}_X(n))$, have $\Gamma_*(X, \mathcal{O}_X(m)) \times \Gamma_*(X, \mathcal{O}_X(n)) \to \Gamma(X, \mathcal{O}_X(n+m))$

making $\Gamma_* \mathcal{O}_X$ a ring.

Have $\underset{\Gamma_* \mathcal{F}_m}{\Gamma(X, \mathcal{F}(m))} \times \underset{\Gamma_* \mathcal{O}_{X,n}}{\Gamma(X, \mathcal{O}_X(n))} \longrightarrow \underset{\Gamma_* \mathcal{F}_{m+n}}{\Gamma(X, \mathcal{F}(n+m))}$.

- $R_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$
$r \longmapsto r/1$

- We describe $\widetilde{\Gamma_* \mathcal{F}}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f))$
q coh $\Rightarrow$ Enough to produce $\Gamma_* \mathcal{F}[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{F})$
Note $\Gamma_* \mathcal{F}|_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{F}(\lambda))$ is
nat a $\Gamma_* \mathcal{O}_{D_+(f)} = \bigoplus_{\lambda \in \mathbb{Z}} \Gamma(D_+(f), \mathcal{O}_X(\lambda))$ mod.
In the later ring the image of f via

$$\begin{array}{ccc} R & \longrightarrow & \Gamma_* \mathcal{O}_X \\ & \searrow & \downarrow \\ \text{graded} & & \Gamma_* \mathcal{O}_{D_+(f)} \end{array}$$

is invertible as $1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

$$\downarrow \qquad\qquad \searrow \qquad \qquad \qquad 1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$$
$$\Gamma_* \mathcal{O}_{D_+(f)}$$

Thus have, graded $R[1/f] \longrightarrow \Gamma_* \mathcal{O}_{D_+(f)}$. This makes $\Gamma_* \mathcal{F}|_{D_+(f)}$ an $R[1/f]$ module and induces a graded R-lin map, $\Gamma_* \mathcal{F}[1/f] \longrightarrow \Gamma_* \mathcal{F}|_{D_+(f)}$

Restrict to the deg 0 piece to obtain the desired map.

Thm Assume R is a finitely generated R_0 alg; i.e $\exists$ $g_1, g_2 \ldots g_r$ +ve deg homo elts s.t

$$R_0[x_1, \ldots x_r] \longrightarrow R \quad , R_0\text{-lin}$$
$$x_i \longmapsto g_i \qquad \text{is sur.}$$

- For $\mathcal{F} \in Qcoh(X)$, the map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$ is an isom.

Pf: Our hypothesis $\Rightarrow$ $R_+ = \underset{\text{ideal}}{\ }(g_1, g_2 \ldots g_r)$

$$\Rightarrow X = \bigcup_{i=1}^{r} D_+(g_i)$$

Replace g_i by $g_i^{\prod_{j\neq i} \deg g_j}$ to assume $\deg g_i$ are all equal $= d$. Note $X = \bigcup_{i=1}^{r} D_+(g_i)$.

Note $\mathcal{O}_X(nd)$ is invertible for $\forall n$.

Since $\mathcal{F}$ is qcoh it is enough to check that the induced map $(\Gamma_* \mathcal{F}[1/g_i])_0 \longrightarrow \Gamma(D_+(g_i), \mathcal{F})$ $(+)$ is an isom $\forall i$

Needed lemma

Lemma: Y quasi compact scheme, $\mathcal{G} \in Qcoh(Y)$, $\mathcal{L}$ invertible, $s \in \Gamma(Y, \mathcal{L})$. Let

$$D_s = \{ y \in Y \mid s_y \notin m_y \mathcal{L}_y \} \quad m_y \text{ is the max ideal of } \mathcal{O}_{Y,y}.$$

(i) D_s is open

(ii) Given $t_1 \in \Gamma(Y, \mathcal{G})$ if $t_1|_{D_s} = 0$,
$\exists$ $n \in \mathbb{N}$ s.t $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Assume intersection of every two open affines have a finite covering by affine opens in Y. [This is immediate if Y is (quasi) separated]

(iii) Assume intersection of every ... covering by affine opens in Y. [This is immediate if Y is noetherian or Y is (quasi) separated]

Given $t \in \Gamma(D_s, \mathcal{G})$, $\exists\ n \in \mathbb{N}$, $\tilde{t} \in \Gamma(Y, \mathcal{G}\otimes\mathcal{L}^n)$

s.t $\underset{\Gamma(D_s, \mathcal{G}\otimes\mathcal{L}^n)}{t \otimes s^n} = \tilde{t}|_{D_s}$

Pf. (i) $\boxed{\text{Quasicompactness is unnecessary}}$. Choose an affine open covering $Y = \bigcup_j U_j$ s.t

$\mathcal{O}_{U_j} \xrightarrow{\sim} \mathcal{L}|_{U_j}$ $\quad 1 \longrightarrow s_j$, $s|_{U_j} = f_j s_j$, $f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$

$D_s \cap U_j = D(f_j) \underset{\text{open}}{\subseteq} U_j$. Thus $D_s \cap U_j$ is open in U_j. $\forall j$. Thus D_s is open in X. $\square$

Since Y is quasi-compact, we choose a finite subcover of U_j $\quad Y = \bigcup_{j=1}^{r} U_j$. Fix this for (ii), (iii)

(ii) $t_1|_{U_j \cap D_s} = 0$, $U_j \cap D_s = D_{U_j}(f_j)$ [This means the basic affine open given by f_j inside U_j]

Since U_j is affine and $\mathcal{G}|_{U_j}$ is qcoh $f_j^{n_j} \cdot t_1 = 0 \in \Gamma(U_j, \mathcal{G}|_{U_j})$ for some n_j

$\Rightarrow$ $t_1 \otimes s^{n_j}|_{U_j} = 0 \in \Gamma(U_j, \mathcal{G}\otimes\mathcal{L}^{n_j})$

Take $n = \max\{n_1, \dots, n_r\}$, Then $t_1 \otimes s^n|_{U_j} = 0 \ \forall j$

$\Rightarrow$ $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G}\otimes\mathcal{L}^n)$

(iii) Since $D_s \cap U_j = D(f_j)$ in U_j and $\mathcal{G}(D_s \cap U_j) = \mathcal{G}(U_j)[1/f_j]$ [$\because \mathcal{G}$ is qcoh]

$\exists$ n_j s.t $f_j^{n_j} \cdot t$ is the restriction of a section in $\Gamma(\mathcal{G}, U_j)$ to $D(f_j) = D_s \cap U_j$. This means $\exists$ $t_j \in \Gamma(U_j, \mathcal{G}\otimes\mathcal{L}^{n_j})$ s.t $t \otimes s^{n_j} = t_j|_{D_s \cap U_j}$

Let $n_0 = \max\{n_1, n_2, \dots, n_r\}$

Set $t'_j = t_j \otimes s^{n_0 - n_j} \in \Gamma(U_j, \mathcal{G}\otimes\mathcal{L}^{n_0})$

On $U_{j_1} \cap U_{j_2} \cap D_s$ $\quad t'_{j_1} = t'_{j_2} = t|_{U_{j_1}\cap U_{j_2}\cap D_s} \otimes s^{n_0}|_{U_{j_1}\cap U_{j_2}\cap D_s}$

By our hypothesis $U_{j_1} \cap U_{j_2} = \bigcup_{\alpha\in I} V_\alpha^{j_1 j_2}$, where $|I| < \infty$.

Using (ii) on each $V_\alpha^{j_1 j_2}$ and finiteness of the covering $U_{j_1} \cap U_{j_2} = \bigcup V_\alpha^{j_1 j_2}$

$\dots t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$ $\quad n_0 + n_{j_1 j_2}$

by ... on each $V_\alpha^{j_1 j_2}$

Using (ii) and finiteness of the covering $U_{j_1} \cap U_{j_2} = \cup V_\alpha$

find $n_{j_1 j_2}$ such that $t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$
$\in \Gamma(U_{j_1} \cap U_{j_2}, \mathcal{G} \otimes \mathcal{L}^{n_0 + n_{j_1 j_2}})$

set $n = \max_{j_1, j_2 \in I} \{ n_0 + n_{j_1 j_2} \}$

Then $t'_j \otimes s^n \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^n) \ \forall j$

and $t'_{j_1} \otimes s^n|_{U_{j_1} \cap U_{j_2}} = t'_{j_2} \otimes s^n|_{U_{j_1} \cap U_{j_2}} \ \forall j_1, j_2$

$\Rightarrow \tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$ s.t $\tilde{t}|_{U_j} = t_j \otimes s^n \ \forall j$

Clearly on $\tilde{t}|_{D_s} = t \otimes s^n$ on $\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Back to (+).

injectivity. If s/g_i^n goes to zero

$s|_{D_+(g_i)} = 0 \Rightarrow \exists n_i$ s.t $s \cdot g_i^{n_i} = 0 \in \Gamma(X, \mathcal{F}(n+d))$

$\Rightarrow s/g_i^n = 0$ in $(\Gamma_* \mathcal{F}[1/g_i])_0$

Surjectivity. Note $X = \cup D_+(g_i)$

$D_+(g_\alpha) \cap D_+(g_\beta) = D_+(g_\alpha g_\beta)$

So we can apply (ii) of Lemma above on X.

Given $t \in \mathcal{F}(D_+(g_i))$. $\exists n_i \in \mathbb{N}$ s.t

$t \otimes g_i^{n_i} = \tilde{t}|_{D_+(g_i)}$ for some $\tilde{t} \in \Gamma(X, \mathcal{F}(n_i d))$

Then $\tilde{t}/g_i^{n_i} \in (\Gamma_* \mathcal{F}[1/g_i])_0$ with its image

in $\mathcal{F}|_{D_+(g_i)}$ being t.

Thm: Assume that $\exists$ homog elts of +ve deg $g_1, g_2 \ldots g_r$

s.t $R = R_0[g_1, g_2 \ldots g_r]$ and $\operatorname{Proj}(R) = X$ is locally noeth. Given $\mathcal{F} \in \mathrm{Coh}(X)$, $\exists$ a finitely gen $M \in \mathrm{Mod}_R^{gr}$

s.t $\tilde{M} \xrightarrow{\sim} \mathcal{F}$

Pf. Choose $n_1, n_2, \ldots n_r$ s.t $\deg g_i^{n_i} = \deg g_j^{n_j} = d \ \forall i, j$

Then $X = \bigcup_{i=1}^{r} D_+(g_i^{n_i})$. So $\mathcal{O}_X(d)$ is invertible.

Realize $g_i^{n_i} \in \Gamma(X, \mathcal{O}_X(d))$. Note $D_{g_i^{n_i}} = D_+(g_i^{n_i})$

$g_i^{n_i}$ is thought of in $\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$

$\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$ mod. By the lemma above, $\exists\, s_1, s_2, \ldots, s_{r_i} \in \Gamma(X, \mathcal{F}(d_i d))$ such that $\{s_\alpha / g_i^{n_i d_i}\}_{\alpha = 1, \ldots, r_i}$ is a set of gen.

By varying i and possibly increasing d_i, we can find $m \in \mathbb{N}$ & finitely many elts $t_1, t_2, \ldots, t_n \in \Gamma(X, \mathcal{F}(dm))$. s.t $\forall i$, $\{t_i / g_i^{n_i m}\}_{i=1,\ldots,n}$ generate the $\Gamma(D_+(g_i^{n_i m}), \mathcal{O}_X)$ mod $\mathcal{F}(D_+(g_i^{n_i}))$.

Let M be the (finitely generated) R-submodule of $\Gamma_* \mathcal{F}$ generated by $t_1, t_2, \ldots, t_n$.

Claim: The inclusion map $M \hookrightarrow \Gamma_* \mathcal{F}$ induces an isom $\widetilde{M} \longrightarrow \widetilde{\Gamma_* \mathcal{F}}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf It's enough to check that for each i the induced map $\Gamma(D_+(g_i^{n_i}), \widetilde{M}) \longrightarrow \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}})$ is an isom.

$$\begin{array}{ccc} \Gamma(D_+(g_i^{n_i}), \widetilde{M}) & \longrightarrow & \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}}) \\ \Uparrow S & & \Uparrow S \\ (M[1/g_i^{n_i}])_0 & \longrightarrow & (\Gamma_* \mathcal{F}[1/g_i^{n_i}])_0 \end{array}$$

The injectivity follows as $M \subseteq \Gamma_* \mathcal{F}$; surjectivity follows from the diag

$$\begin{tikzcd} (M[1/g_i^{n_i}])_0 \arrow[r] \arrow[dr] & \mathcal{F}(D_+(g_i^{n_i m})) \\ & (\Gamma_* \mathcal{F}[1/g_i^{n_i m}])_0 \arrow[u, "\wr"'] \end{tikzcd}$$

where the top arrow is surj by construction.

Invertible sheaves:

Def: X be a scheme. A locally free sheaf of rk 1 is called an invertible sheaf.

Prop: i) Let $\mathcal{L}$ be an invertible sheaf. Then

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{O}_X \qquad (\varphi \otimes s) \mapsto \varphi(s)$$

is an isom

ii) For an $\mathcal{O}_X$-mod $\mathcal{F}$, suppose there is an $\mathcal{O}_X$-mod $\mathcal{G}$ and an isom $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{O}_X$. Then $\mathcal{F}$ is invertible.

Pf (i) Given $x \in X$, choose an open nbhd U of x s.t $\mathcal{L}|_U \xleftarrow{\sim} \mathcal{O}_U$. We have a diag using this isom

$$\begin{tikzcd}
\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)|_U \otimes_{\mathcal{O}_U} \mathcal{L}|_U \arrow[r] & \mathcal{O}_U \arrow[d, equal, "id"] \\
\mathcal{O}_U \xrightarrow{\sim} \underline{\mathcal{H}om}_{\mathcal{O}_U}(\mathcal{O}_U,\mathcal{O}_U) \otimes \mathcal{O}_U \arrow[r] \arrow[u, "\wr"] & \mathcal{O}_U
\end{tikzcd}$$

Since the bottom row is an isom, we are done.

(ii) Check at stalks.

Prop. X be a scheme. The isom classes of invertible $\mathcal{O}_X$-mods form an abelian group under $\otimes$ operation, denoted $\text{Pic}(X)$ - called the Picard group or the group of invertible sheaf.

Prop\Eg. • R be an $\mathbb{N}$-graded ring. Suppose $\exists\, g_1, g_2, \dots g_r$ each of deg d such that $\text{Proj}(R) = \bigcup_{i=1}^{r} D_+(g_i)$.

Then for each n, $\mathcal{O}_X(nd)$ is invertible.

- So if $d=1$, each $\mathcal{O}_X(n)$ invertible
- Assume R is gen over R_0 by deg 1 elt as an alg (i.e R standard graded, then $d=1$ and each $\mathcal{O}_X(n)$ is invertible.

Eg: We will see $\text{Pic}(\mathbb{A}^n_{\mathbb{C}}) \simeq \{id\}$, $\text{Pic}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\sim} \mathbb{Z}\cdot\mathcal{O}(1)$

$\mathbb{P}^n_{\mathbb{C}} = \text{Proj}(\mathbb{C}[x_0, \dots x_n])$

End of 13.11.24 Lecture.

Def. X scheme, $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$ is called <u>globally generated</u> if there is a surjection of $\mathcal{O}_X$ mods $\bigoplus_I \mathcal{O}_X \longrightarrow \mathcal{F}$, ($|I|$ need not be finite)

Rmk. Note $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \xleftarrow{\sim} \Gamma(X, \mathcal{G})$. So giving a sure morphism of $\mathcal{O}_X$-mod $\bigoplus_I \mathcal{O}_X \to \mathcal{G}$ is the same as choosing $|I|$ many elts of $\Gamma(X,\mathcal{G})$, such that those generate every stalk $\mathcal{G}_x$, $x \in X$.

Def: An invertible $\mathcal{O}_X$-mod $\mathcal{L}$ is called ample, if for any $\mathcal{F} \in \text{Coh}(X)$ $\exists\, n_{\mathcal{F}} \in \mathbb{N}$ s.t $\forall\, n \geq n_{\mathcal{F}}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is globally generated.

Divisors

Wednesday, November 13, 2024 9:11 AM

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